# CIS7 Chapter 8 Counting Notes

**Combinatorics** is the branch of mathematics largely concerned with counting discrete objects, and counting plays a central role in many aspects of computer science.

Many problems in computer science involve **counting** possible solutions, **enumerating** those solutions, or **finding** the optimal solution among a candidate set.

In order to count or enumerate a collection of discrete objects, one can consider the choices that need to be made in order to generate the objects in the collection, and then count the number of objects which can be generated based on those choices and a few basic rules.

Example:

1. Alice order **soup or sandwich**: **6 types of soups, 12 different sandwiches**.

Alice must choose **either one of six soups or one of 12 sandwiches** for a total of **6 + 12 = 18 different meals**

Alice’s meal choice problem generalizes to the ***Sum Rule***.

**Sum Rule**: If A and B are **disjoint finite sets** then the number of ways of choosing **a single element** from **A or B is |A ∪ B| = |A| + |B|.**

**Sum Rule:** the sum rule is applicable when one must make one choice from the union of two (or more) sets of alternatives; sum rule can be generalized to more than two sets of alternatives in an obvious way.

General Sum Rule: If A1, A2, . . . , An are mutually disjoint finite sets then the number of ways of choosing a single element from A1, A2, . . . , or An is |A1 ∪ A2 ∪ · · · ∪ An| =|A1| + |A2| + · · · + |An|.

1. Bob orders **both soup and sandwich**: **6 types of soups, 12 different sandwiches**.

Bob must choose **one soup** and one sandwich. He has **6 ways to choose a soup and 12 ways to choose a sandwich**.

Since any soup can be paired with any sandwich, there are **6 · 12** = **72 total different meals that he can order.**

If in addition Bob were to choose a drink and there were eight possible drinks, Bob would then have **6 × 12 × 8 = 576** possible meal choices. The product rule can clearly be extended to this more general case.

Bob’s meal choice problem generalizes to the ***Product Rule*.**

**Product Rule:** If A and B are **finite sets** then the number of ways of choosing an element from **A and an element from B** is **|A × B| = |A| × |B|.**

The **General Product Rule** If A1, A2, . . . , An are ***finite sets***, then the number of ways of choosing an element from A1, an element from A2, . . . , and an element from An is |A1 × A2 ×· · · × An| = |A1| × |A2| × · · · × |An|.

The **product rule** is applicable when **one must make two (or more) consecutive choices from sets** **of alternatives.**

**Example 8.1**

Personal Identification Numbers or PINs are entered on a numeric keypad and, hence made up entirely of digits. The PINs on our office locks are required to be exactly **4 digits**. How many different PINs are possible?

**The set of digits, D = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 has cardinality 10**. Each PIN corresponds to an element of **D × D × D × D**. There are **104 = 10,000** different PINs.

1. How many different 7 digit PINs are there?

**107 = 10,000,000**.

1. How many different 4 to 7 digit PINs are there?

A single PIN has either 4 or 5 or 6 or 7 digits. We use the product rule to separately count the sets of 4-digit, 5-digit, 6-digit, and 7-digit passwords then use the sum rule to count the union of these sets. The number of 4 to 7 digit pins is **104 + 105 +106 + 107 = 11,110,000**.

**Example 8.2**

Passwords are often composed of alpha-numeric characters, a, b, ..., z, 0, 1, 2, ..., 9 on systems that are not case-sensitive or A, B, ..., Z, a, b, ..., z, 0, 1, 2, ..., 9 on systems that are case-sensitive.

1. How many **4-character alpha-numeric passwords** are there if you can use **upper- and lower-case letters and digits** (i.e. case-sensitive)?

There are **26 upper-case letters, 26 lower-case letters, and 10 digits for a character**

Set C of size **62** = (**26+26+10**). The total number of possible passwords is

**|C × C × C × C| = (62)4 = 14,776,336.**

B. If a hacker has code that can try out passwords on a system at a rate of **1 per second**, how long would it take her to break into a system that

1. Uses **4-digit passwords**? (numeric: 0-9)

**104 = 10,000 seconds = 2 hours 46 minutes 40 seconds.**

1. Uses 4-character **case-sensitive, alpha-numeric** passwords?

**14,776,336 seconds = 171 days 32 minutes 16 seconds**

**Example 8.3**

**Bit strings** are strings composed of **0s and 1s.**

1. How many **bit strings** are there with **8 bits**?

**28 = 256.**

1. How many **bit strings** are there with **16 bits**?

**216 = 65,536.**

1. What is **the largest integer** that can be represented in **16-bit two’s complement**?

Since **positive integers** **in two’s complement must have a 0 in the leftmost position**, we have only **15 places to represent the magnitude of the integer**. The largest integer we can represent is **0111 1111 1111 1111 = 215 − 1 = 32767.**

**Example 8.4**

Picking Students:

Suppose that there are **three sections** of a discrete structures **class containing 73, 64, and 41 students**, respectively.

1. How many distinct ways are there of **choosing one discrete structures student** to write up a sheet of notes for everyone to use at the final?

**The sum rule applies**, yielding **73 + 64 + 41 = 178 possibilities**.

1. How many distinct ways are there of **choosing one discrete structures student from each class** to form an advisory committee?

**The product rule applies**, yielding **73 · 64 · 41 = 191,552**.

1. How many distinct ways are there of **listing six different discrete structures students from a 41 person section** to go to the board **one after the other to present problem** solutions?

Consider the choices that must be made in generating such a section. There are **41 students** **to choose from** **as the first presenter**, but then there are only **40 students to choose from as the second presenter**, 39 as the third presenter and so on. Note that multiple choices must be made (so the product rule applies) and the size of the sets of alternatives (from which these choices must be made) shrinks with each successive choice. The result is **41 · 40 · 39 · 38 · 37 · 36 = 3,237,399,360.**

This is an example of **permutations**, which is discussed in more detail in Section 8.4.

1. How many distinct ways are there of choosing **six discrete structures students to form the course volleyball team**?

Now we have to **choose a set of** **six students out of the 178 discrete structures students**. If we count the ways to make a list of 6 students, as in c, we get **178 · 177 · 176 · 175 · 174 · 173** **possible ordered lists of 6 students**.

**Each set of 6 students appears** in this list **6! = 6 · 5 · 4 · 3 · 2 · 1 times** so the number of sets is of six students out of the 178 is

This is an example of **combinations**, which is discussed in more detail in Section 8.5.

**Example 8.5**

More Passwords Suppose passwords are restricted to **6 case-sensitive alpha-numeric characters** and **must contain at least 1 digit and at least 1 letter**. How many are there?

There are **(62)6 passwords** composed of **6 case-sensitive alpha-numeric characters** with no other restrictions. Of these, **(52)6 are composed of letters only** and **(10)6 are composed of digits only**. **All the others have at least one digit and at least one letter**. So the answer is [(62)6 − (52)6 − (10)6].

Suppose passwords may have **6 to 10 case-sensitive alpha-numeric characters** and **must contain at least 1 digit and at least 1 letter**. How many are there?

Since a password may have **6 or 7 or 8 or 9 or 10 letters**, we can **count each of these possibilities separately and apply the sum rule** to get the result. [(62)6−(52)6−(10)6]+[(62)7−(52)7−(10)7]+[(62)8−(52)8−(10)8]+[(62)9−(52)9−(10)9]+[(62)10−(52)10−(10)10]

## Inclusion-Exclusion Principle

In our description of the **sum rule** above, we assumed that ***the sets of alternatives from which one must make a single choice*** were **mutually disjoint**, i.e., that ***they did not share any common elements***.

In situations when a **single choice must be made from sets of alternatives** **which are not disjoint.**

For example, suppose that Alice and Bob’s deli has both a lunch menu and a dinner menu, and that one is allowed to order off either menu at lunchtime. Furthermore, suppose that some sandwiches that are available for lunch are also available for dinner? In how many ways can Bob pick his sandwich? In mathematical terms, let Sl denote the set of lunch sandwiches, and let Sd denote the set of dinner sandwiches. Applying the sum rule, the total number of sandwich choices is **|Sl ∪ Sd|** which is **|Sl|+|Sd|** if the sets **Sl and Sd are disjoint**. If **Sl and Sd are not disjoint**, then **|Sl|+|Sd|** will double-count those sandwiches which are in both Sl and Sd, i.e., available for both lunch and dinner. To **avoid this double counting, we must subtract a quantity that corresponds to the number of sandwiches which are double-counted**; this is precisely **|Sl ∩ Sd|**

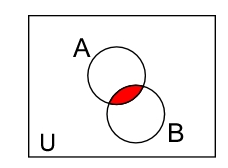


Figure 8.1: |A|+|B| counts the elements in A and the elements in B, but **the elements of A∩B (the red/shaded ones) are counted twice**. The proper total count is **|A∪B| = |A|+|B|−|A∩B|** as dictated by the **inclusion-exclusion principle**.

**Inclusion-exclusion Principle:**  If A and B are finite sets then **|A∪B| = |A|+|B| −|A∩B|.**

**Example 8.6**

How many strings of **6 upper-case letters start with A or end with Z**?

**265 start with A.**

**265end with Z.**

**264 start with A and end with Z and they were counted twice**. The answer is **265 + 265 − 264**

**Example 8.7**

This problem is Exercise 20 in Kenneth H. Rosen, Discrete Mathematics and Its Applications [7, page 311].

How many **positive integers between 1000 and 9999 inclusive** (There are 9000 consecutive integers = (9999-1000)+1)

1. Are **divisible by 9**?

Every **9th integer is divisible by 9** and **9 divides 9000 evenly so there are 1000**.

1. Are **even**?

Half the integers are even and **2 divides 9000 evenly so 4500 are even**.

1. Have distinct digits?

Integers between 1000 and 9999 all have 4 digits and the **leftmost digit cannot be 0 so there are 9 choices**. The **second digit can be any digit but the first so there are 9 choices**. There are **8 choices for the third digit and 7 for the fourth**.

By the Product Rule, **9 × 9 × 8 × 7 = 4536** have distinct digits.

1. Are **not divisible by 3**?

**Every 3rd integer is divisible by 3** and **3 divides 9000 evenly so there are 3000** **that are divisible by 3** and **9000 − 3000 = 6000 that are not divisible by 3**.

1. Are **divisible by 5 or 7**? Use the Inclusion-Exclusion Principle here.

We compute **the number divisible by 5 plus the number divisible by 7 minus the number divisible by both** (i.e. divisible by **35)**.

* **Divisible by 5: 9000/5 = 1800**
* **Divisible by 7: 9000/7 = 1285.71429 . . . so there are either 1285 or 1286**.

This take some extra checking, e.g. there are 3 integers from 4 to 12 that are

* Divisible by **4 (4, 8, 12) but only 2 integers from 3 to 11 (4, 8) that are divisible by 4**. The range in each case includes 9 integers and **9/4 = 2.2**5.

The first integer starting at **1000** that is **divisible by 7 is 1001** and **1001 + 7** ·

**1285 = 1001 + 8995 = 9996** so the answer is 1286.

* Divisible by 35: **9000/35 = 257.142857** . . .. The first integer starting at **1000** that is divisible by 35 is 1015 and **1015+ 35 · 257 = 10010 > 9999** so the answer is **257** (not 258).

• Now we apply the Inclusion-Exclusion Principle. The number divisible by 5 plus the number divisible by 7 minus the number divisible by both is **1800 + 1286 − 257 = 2829**.

1. Are not divisible by either 5 or 7?

This is **9000 minus the number that are divisible by 5 or 7**, **9000 − 2829 = 6171**.

1. Are divisible by 5 but not by 7?

This is the number divisible by **5 but not divisible by 35, 1800 − 257 = 1543**.

1. Are divisible by 5 and 7?

This is the same as the number divisible by **35 which is 257**

## Pigeonhole Principle

**Pigeonhole Principle** If **k + 1 or more objects are placed in k boxes**, then there must exist a box that **contains two or more objects**.

**For example**: if **13 or more people gather in a room**, then it is guaranteed that **at least two of them share the same birth month**. While this may happen in a gathering of 12 or fewer people, it is not guaranteed.

**Example 8.8**

Suppose that there are **102 students in two sections of discrete structures**. If they all take the final, will **at least two of them get the same grade**?

There are **101 possible grades** **0, 1, . . . , 100** so the result follows from the pigeonhole principle.

**Example 8.9**

If I use the last **two digits of their social security numbers as a code to post grades in anonymity**, will at **least two students get the same code**?

There are **100 2-digit codes**, **00 through 99** so by the time I **list the first 101 students there will be two with the same code.**

**Example 8.10**

If a drawer contains **12 red socks and 12 blue socks** and I pull some socks out in the dark,

1. How many must I pull out to be sure of having a **pair**?

**3** because there are only **2 colors**.

1. How many must I pull out to be sure of having a **pair of red socks**?

**14** because I might pull out **all the blue ones and just 2 red ones**.

1. How many must I pull out to be sure of having **at least one of each color**?

**13** because I might pull out all **12 of one color and just one of the other color**.

**Example 8.11**

**Every integer n** has a multiple that has only 0s and 1s in its decimal expansion.

**Proof**

Consider the n + 1 numbers 1, 11, . . . , 11111...11 where the last number has (n + 1) ones. If we evaluate each of these numbers **mod n**, two of them must give the same value as there are only n possible results, 0, . . . , n − 1. If then a − b is divisible by n. So take the two numbers that result in the same value and subtract the smaller from the larger. The result is a multiple of n and has only 0s and 1s in its decimal expansion.

Here’s an example. Take n = 6.

1 mod 6 = 1

11 mod 6 = 5

111 mod 6 = 3

1111 mod 6 = 1

so 1111 − 1 = 1110 is a multiple of 6.

**Example 8.12**

In any set of **n + 1** **positive integers** **not exceeding 2n**, there must be one integer that divides another.

Write each of the n + 1 integers as a power of 2 times an odd integer, aj = 2ej qj .

Then q1 . . . qn+1 are **n+ 1 odd integers less than 2n**. Two of them must be the same.

One of the corresponding aj s divides the other.

### Generalized Pigeonhole Principle

Suppose that {x1, x2, . . . , xn} is a collection of n numbers, and let x be the average of these numbers, i.e.,

Pigeonhole formula. See formula in textbook chapter 8.

Then it clearly cannot be the case that every number is less than the average; in other words, there must **exist at least one number that is at least as large as the average**. Returning to our pigeonhole example, suppose that 21 pigeons attempt to roost in 10 locations. The average number of pigeons per location is 21/10 = 2.1, and by our argument above, it cannot be the case that every location contains fewer than the average number of pigeons. Thus, at least one location must contain at least 2.1 pigeons. Since pigeons are discrete objects (we can’t have 1/10 of a pigeon!), there must exist a location with d2.1e = 3 pigeons. This is the essence of the generalized pigeonhole principle.

**Generalized Pigeonhole**: Principle If N objects are placed in k boxes, then there must exist a box that contains [N/k] or more objects.

Example 8.13

Suppose that there are **51 students** in a discrete structures class. How many students are guaranteed to **have birthdays in the same month**? (In particular, what is the largest number of students that are guaranteed to have the same birth month?)

51/12 = 5 students are guaranteed to have birthday the same month.

## Permutation

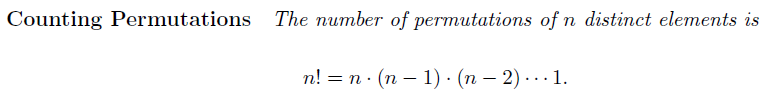
A **permutation** is an **ordered arrangement of a set (or subset) of objects**.

For example, that one had to run four errands, a trip each to (1) the grocery store, (2) the dry cleaners, (3) the hardware store, and (4) the post oﬃce. One would have to decide in which order these errands would be performed. Any such ordering of these errands is a permutation.

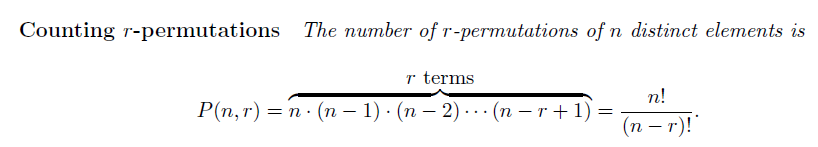
***Each permutation may have an associated cost*** (e.g., the total driving distance if the errands are processed in a speciﬁc order), and one often wants to minimize this cost. A brute force approach would be to list all possible permutations, determine each associated cost, and pick the permutation with the least cost.

How many such permutations are there? One could choose any of the ***four errands to start with***, then ***any of the three remaining errands second, then either of the two remaining errands third, and ﬁnally the last remaining errand fourth***. Note that these choices are made consecutively, so the **Product Rule applies**: There are four possibilities for the ﬁrst choice, three for the second, two for the third, and one choice for the last, for a total of 4·3·2·1 = 4! = 24 total possibilities (permutations). If there were n errands to run (or n objects to order), the number of possible permutations is n!.

**Counting Permutations:**



Now suppose that one only had time to run **two of the four errands**. How many ordered arrangements of two of the four errands are there? Applying our analysis from above, there are four choices for the ﬁrst errand and three remaining choices for the second, yielding ***4·3 = 12 possible arrangements***. More generally, an ordered arrangement of **r objects** from a collection is referred to as an **r-permutation** and denoted by **P(n,r)**; ***calculators often use the notation nPr***. Applying the above analysis to the general case yields the nmber of such arrangements.



Note that **r-permutations** are a generalization of permutations; indeed, and **n-permutation** is an ordered arrangement of all n out of n objects, and the number of such n-permutations is **P(n,n) = n!/0! = n!.**

**Example 8.14**

A wedding party consists of the bride, the groom, the bride’s mother and father, the groom’s mother and father, the best man, the maid of honor, two ushers, and two bride’s maids.

1. How many ways are there of **arranging all of them in a row** for a picture?

There are 12 people in the wedding party so there are P(12,12) = 12! ways of arranging them in

a row.

1. How many ways if the **bride and groom stand together on the left side** of the line?

There are **2 ways to arrange the bride and groom on the far left side** of the line and P(10,10) =10! ways of arranging the rest of the party so 2·10! possible arrangements.

1. How many ways if the **bride and groom are together but anywhere in the line**?

There are P(10,10) = **10!** ways of **arranging the rest of the party without the bride and groom**. Then the **bride and groom together can be place between and two of the lined up people** or to **the left or to the right of all of them**. That’s **11 diﬀerent positions**. There are 2 ways to arrange the bride and groom. The total number of arrangements is 10!·11·2.

1. How many ways can **5 members of the wedding party line up** for a picture?

We must line up 5 people out of 12 so P(12,5) = **12·11·10·9·8**.

**Example 8.15:**

On the trip I am about to take, I must visit Florence, Milan, Venice, London, Bristol, and Warwick.

1. How many diﬀerent itineraries are possible?

There are 6 cities so 6! possible itineraries.

1. How many itineraries are possible if all the British cities are consecutive and all the Italian cities are consecutive?

Florence, Milan, and Venice are in Italy. London, Bristol, and Warwick are in England. There are 3! orders for the Italian cities and 3! orders for the British cities. I can go to Italy ﬁrst or to England ﬁrst so there are 3!·3!·2 = 72 possible itineraries.

**Example 8.16**

1. How many permutation are there of the letters A B C D E F G H I J?

P(10,10) = 10!.

1. How many of them contain the block
2. HEAD

First arrange the other letters, B, C, F, G, I, J. There are 6! arrangements. Then place the block HEAD between two of the arranged letters or at one of the ends. There are 7 places it can go. That makes a total of 7·6! strings. You can also think of this as gluing the H,E,A,D together to form one block and then counting all the arrangements of the 7 blocks, HEAD, B, C, F, G, I, J to get 7!.

1. HJF

This is just like part i but you must arrange the other 7 letters and then there are 8 places where HJF can go.

1. BIGFACEDHJ

Just one.